Lecture 5

Quantifiers (contd.), Well-formed Formulas, Logical Equivalence of WFFs

Truth Values of Quantifiers

Let $P(x)$ be a propositional function and domain $= \{x_1, x_2, \ldots, x_n, \ldots\}$:

- $\forall x P(x)$ is **true** if $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n) \wedge \ldots$ is true, is **false** otherwise.
- $\exists x P(x)$ is **true** if $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n) \vee \ldots$ is true, is **false** otherwise.

Well-formed Formulas

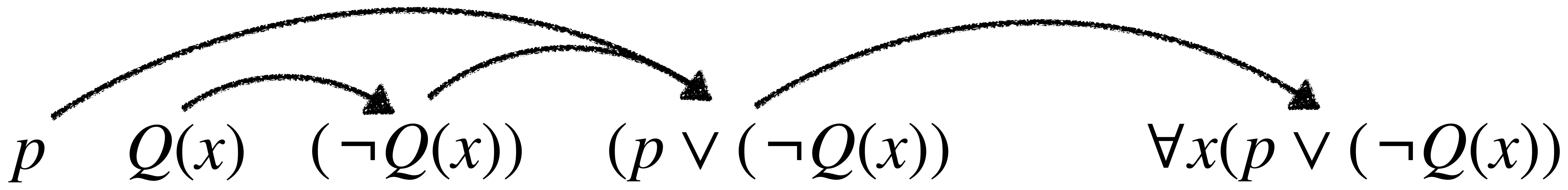
Definition*: 1. Propositional variables such as $p, q, r,$ etc., are WFFs.

2. Propositional functions such as $P(x), Q(x, y), R(x, y, z),$ etc., are WFF.

3. If α and β are two WFFs, then $(\neg\alpha), (\alpha \vee \beta), (\alpha \wedge \beta), (\alpha \rightarrow \beta),$ and $(\alpha \leftrightarrow \beta)$ are WFFs.

4. If α is a WFF where all occurrences of x are free, then $\forall x\alpha$ and $\exists x\alpha$ are WFFs.

Example: $\forall x(p \vee (\neg Q(x)))$ is a WFF.



$\forall x(\exists x(P(x)))$ is not a WFF.

**WFF in FOL are slightly different from how they are defined here.*

Precedence of Quantifier

$\forall$ and $\exists$ have higher precedence than all the logical operators.

$\forall x P(x) \vee Q(x)$ is $(\forall x P(x)) \vee Q(x)$ not $\forall x (P(x) \vee Q(x))$

$\forall x P(x) \rightarrow \exists y Q(y)$ is $(\forall x P(x)) \rightarrow (\exists y Q(y))$ not $\forall x (P(x) \rightarrow \exists y Q(y))$

Binding Variables

- An occurrence of a variable that is not bound by a quantifier is called **free**.
- An occurrence of a variable that is bound by a quantifier is called **bound**.

Example:

$$\forall y \exists x (x + y) = 1$$

Not a proposition, both x & y are free.

Not a proposition, x is bound but y is free.

A proposition as no variable is free.

Scope

- The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.

Example:

$$\exists x\,(P(x) \land Q(x)) \lor \forall x\, R(x)$$

Scope of $\exists x$ is $P(x) \land Q(x)$

Scope of $\forall x$ is $R(x)$

Tip: To avoid confusion, use different variables:

$$\exists x\,(P(x) \land Q(x)) \lor \forall y\, R(y)$$

Logical Equivalence of WFFs

Definition: Two WFFs containing same set of propositional variables and propositional functions *with no free variables* are **logically equivalent** if and only if they have the same truth value, no matter which **propositions** and **predicates** are substituted in WFFs and which **domain** is used for variables.

Example: $\forall x(P(x) \wedge Q(x))$ is logically equivalent to $\forall x P(x) \wedge \forall x Q(x)$

Check it in the book if you are interested.

Negating Quantified Expressions

$q =$ "Every student in this class has taken ICS."

$q = \forall x P(x)$, where $P(x)$ = "x has taken ICS" and domain is the set of students in this class.

What's the negation of q?

$\neg q$ = "It is not the case that every student in this class has taken ICS."

= "There is a student in this class who has not taken ICS."

= $\exists x \neg P(x)$

This illustrates the logical equivalence:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Negating Quantified Expressions

$q =$ "There's a student in this class who got A^* in ICS."

$q = \exists x P(x)$, where $P(x)$ = "x got A^* in ICS" and domain is set of students in this class.

What's the negation of q?

$\neg q$ = "It is not the case that there's a student in this class who got A^* in ICS."

= "Every student in this class did not get A^* in ICS."

= $\forall x \neg P(x)$

This illustrates the logical equivalence:

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Ordering Matters

Consider the following statement with domains as set of integers

$$\forall y \exists x (x + y) = 1$$

It corresponds to "For every y there exists an x such that $(x + y) = 1$". The statement is **true**.

Consider a similar statement with domains as set of integers

$$\exists x \forall y (x + y) = 1$$

It corresponds to "There exists an x such that for every y $(x + y) = 1$". The statement is **false**.

Evaluating Nesting Quantifiers

How to formally evaluate a WFF with nesting quantifiers?

Example: $\forall y \exists x (x + y) = 1$

Let $Q(y) = \exists x (x + y) = 1$

$= \text{There exists an } x \text{ such that } x + y = 1$

Then, $\forall y Q(y) = \forall y \exists x (x + y) = 1$

$\forall y Q(y)$ is true if $\ldots Q(-2) \wedge Q(-1) \wedge Q(0) \wedge Q(1) \wedge Q(2) \ldots$ is true and is false otherwise.